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NEW INTEGRAL TRANSFORMS IN BOUNDARY-VALUE PROBLEMS FOR LAMINATED COMPOSITES WITH A PERIODIC STRUCTURE†

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An elliptic equation for a layered regularly inhomogeneous (composite) material serves as an example for the introduction of new integral transforms that enable boundary-value problems to be solved in quadratures, without the need to solve boundary-layer problems. These integral transforms are used to solve simple boundary-value problems for a layered composite and to obtain a fundamental solution of an elliptic equation in an infinite two-dimensional laminar medium.

THE DIFFICULTY of solving boundary-value problems for strongly inhomogeneous (composite) materials is due to the rapid oscillation of the coefficients of the equations. For materials of periodic structure these coefficients are periodic functions. If the period is small compared with the characteristic dimensions of the problem, asymptotic averaging may be applicable [1–3]. This method yields an asymptotically correct approximation to the exact solution (for small values of the structure period ε), based on solving an averaged problem for a homogeneous (or rather, homogenized) material and "local" problems over one period. Far from the boundaries of the domain, averaging gives a good approximation to the exact solution even in the zeroth approximation (see [1, 2]). Near the boundary, however (i.e. at distances commensurate with ε), the use of this method is fraught with difficulties.

The boundary layer method [1] can be used to find an asymptotic solution for the problem near the boundary (see also [4]). The boundary layer method has been applied [5, 6] to the problem of a macrocrack in the periodically structured composite material. However, the use of the method involves the need to solve boundary layer problems (see [1, 4–6]), which are considerably more difficult than local problems. For example, in the simplest case of a layered composite, the local problems can be solved exactly [1, 2], but the boundary layer problems are amenable to numerical solution only (see [6]).

1. Consider the following boundary-value problem for a periodically structured layered composite, defined in the upper half-plane $x_2>0$ (see Fig. 1):

$$\frac{\partial}{\partial x_1}\left(\lambda_1^{(\varepsilon)}(x_1)\frac{\partial u}{\partial x_1}\right)+\lambda_2^{(\varepsilon)}(x_1)\frac{\partial^2 u}{\partial x_2^2}=0 \tag{1.1}$$

$$u\,|_{x_2=0}=g(x_1) \tag{1.2}$$

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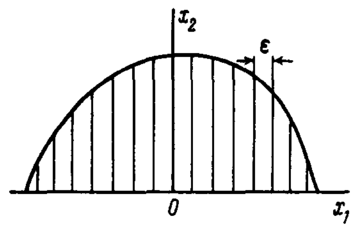

FIG. 1.

Instead of the Dirichlet condition (1.2) at $x_2 = 0$ there may be a Neumann condition

$$\lambda_2^{(\varepsilon)}(x_1)\, \partial u/\partial x_2 \,|_{x_2=0} = -q(x_1) \tag{1.3}$$

It is then assumed that

$$\int_{-\infty}^{+\infty} q(x_1)\, dx_1 = 0.$$

Let us assume that the coefficients of Eq. (1.1) are rapidly oscillating ε-periodic functions of x_1 ($\varepsilon \ll 1$). Set

$$\lambda_\alpha^{(\varepsilon)}(x_1) = \lambda_\alpha(y), \quad y = x_1/\varepsilon, \quad \alpha = 1, 2$$

where $\lambda_\alpha(y)$ are singly periodic functions of y. We shall also assume that $\lambda_\alpha(y)$ are piecewise smooth functions, with discontinuities of the first kind on the contact line(s) of the different constituents of the composite; at the points of discontinuity of the coefficients certain matching conditions, corresponding to ideal contact, are satisfied:

$$[u] = 0, \quad [\lambda_1 \partial u/\partial x_1] = 0 \tag{1.4}$$

These problems model the steady temperature distribution or antiplane elastic stress–strain state of a layered composite material of periodic structure.

2. Before attempting to solve such problems analytically, we have to consider the following auxiliary problem: it is required to expand a piecewise-smooth function $f(x)$ in terms of solutions of the equation

$$(A^{(\varepsilon)}(x)\, z')' + \mu^2 \rho^{(\varepsilon)}(x)\, z = 0, \quad 0 < x < \infty \tag{2.1}$$

$$A^{(\varepsilon)}(x) = A(y), \quad \rho^{(\varepsilon)}(x) = \rho(y)$$

where $y = x/\varepsilon$ and $A(y)$ and $\rho(y)$ are singly periodic, piecewise-smooth functions of y; the prime denotes differentiation with respect to x.

To solve the auxiliary problem we use the method proposed in [7], considering the problem

$$\frac{\partial}{\partial x}\left(A^{(\varepsilon)}(x) \frac{\partial u}{\partial x}\right) = \rho^{(\varepsilon)}(x) \frac{\partial u}{\partial t} \tag{2.2}$$

$$u\,|_{t=0} = f(x) \tag{2.3}$$

Using the Laplace transformation

$$z(x, p) = \int_0^\infty u(x, t)\, e^{-pt}\, dt \tag{2.4}$$

we deduce from (2.2) and (2.3) that

$$(A^{(\varepsilon)}(x)\, z')' - p\rho^{(\varepsilon)}(x)\, z = -\rho^{(\varepsilon)}(x)\, f(x), \quad 0 < x < \infty \tag{2.5}$$

The solution of the homogeneous equation with singly periodic coefficients

$$(A(y)\, z')' - p\rho(y)\, z = 0 \tag{2.6}$$

will be sought as a double-scaled asymptotic expansion [1–3];

$$z = z_0(x) + \sum_{k=1}^{\infty} \varepsilon^k z_k(x, y) \tag{2.7}$$

where $z_k(x, y)$ ($k = 1, 2, 3, \ldots$) are singly periodic functions of y. An application of the asymptotic averaging procedure [1–3] yields a proof of the following lemma.

Lemma 1. The full asymptotic expansion (2.7) for a solution of Eq. (2.6) has the form

$$z = z_0(x) + \sum_{k=1}^{\infty} \varepsilon^k N_k(y) \frac{d^k z_0(x)}{dx^k} \tag{2.8}$$

where $z_0(x)$ is a solution of the equation

$$z_0''(x) - p\varkappa^2 z_0(x) = 0, \quad \varkappa^2 = \langle A^{-1}\rangle \langle \rho \rangle \tag{2.9}$$

and $N_k(y)$ are singly periodic functions of y which are solutions of the recursive chain of local problems

$$\begin{gathered}\frac{d}{dy}\left(A(y)\frac{dN_k(y)}{dy}\right) = -\frac{d}{dy}(A(y)N_{k-1}(y)) - \\ - A(y)\frac{dN_{k-1}(y)}{dy} - A(y)N_{k-2}(y) + \varkappa^{-2}N_{k-2}(y)\rho(y), \quad k = 1, 2, 3, \ldots \\ N_{-1}(y) \equiv 0, \quad N_0(y) \equiv 1\end{gathered} \tag{2.10}$$

with matching conditions at the points of discontinuity of $A(y)$ and $\rho(y)$:

$$[N_k] = 0, \quad [A\,(dN_k/dy + N_{k-1})] = 0 \tag{2.11}$$

These functions $N_k(y)$ are determined by solving problems (2.10) and (2.11), apart from constant terms $N_{k0} = N_k(0)$, which may be determined by imposing additional conditions on $N_k(y)$.

The asymptotic expansion (2.8) for solutions of Eq. (2.6) may be justified rigorously by standard techniques [4]. Let $z^{(m)}$ denote the partial sums of the series (2.8):

$$z^{(m)} = \sum_{k=0}^{m+1} \varepsilon^k N_k(y) \frac{d^k z_0}{dx^k}$$

Substituting $z^{(m)}$ into Eq. (2.6) and using (2.9) and (2.10), we obtain an expression for the truncation error:

$$\begin{gathered}P(z - z^{(m)}) = \varepsilon^m (AN'_{m+2})' \frac{d^{m+2} z_0}{dx^{m+2}} + \\ + \varepsilon^{m+1}[(AN'_{m+3} + AN_{m+2})' + AN'_{m+2}] \frac{d^{m+3} z_0}{dx^{m+3}}\end{gathered}$$

Here P is the operator on the left of Eq. (2.6), and the prime denotes ordinary differentiation with respect to y.

If Eq. (2.6) has smooth coefficients and the boundary conditions for $z(x)$ and $z_0(x)$ are identical over some interval $[0, l]$ (this may be ensured by proper choice of the constant terms N_{k0}), then the generalized maximum principle for solutions of differential equations implies the estimate

$$\| z - z^{(m)} \|_{C[0,l]} = O(\varepsilon^m)$$

If the coefficients of Eq. (2.6) are merely piecewise-smooth and one has matching conditions of type (1.4) and accordingly also (2.11) in the formulation of the local problems, then, besides the theoretical justification of (2.8), one must also verify that the asymptotic solution $z^{(m)}$ satisfies the matching conditions with a fairly high degree of accuracy with respect to ε. This may be done as shown in [1, pp. 49–52].

The reader should note that the averaging in (2.9) and later is done using the rule

$$\langle \rho \rangle = \int_0^1 \rho(y)\, dy.$$

Proceeding now to treat the non-homogeneous equation (2.5), we choose the following functions as linearly independent particular solutions of Eq. (2.9):

$$z_0^{(1)}(x, p) = \operatorname{ch}(\sqrt{p}\varkappa x), \quad z_0^{(2)}(x, p) = e^{\sqrt{p}\varkappa x} \tag{2.12}$$

We can then prove the following lemma.

Lemma 2. If the functions (2.12) are taken as the linearly independent particular solutions of Eq. (2.9), one obtains the following linearly independent solutions of Eq. (2.6) from (2.8):

$$\begin{aligned} z^{(1)} &= \Big(\sum_{k=0}^{\infty} \varepsilon^{2k} p^k \varkappa^{2k} N_{2k}(y)\Big) \operatorname{ch}(\sqrt{p}\varkappa x) + \\ &+ \Big(\sum_{k=0}^{\infty} \varepsilon^{2k+1} p^{k+1/2} \varkappa^{2k+1} N_{2k+1}(y)\Big) \operatorname{sh}(\sqrt{p}\varkappa x) \\ z^{(2)} &= \Big(\sum_{k=0}^{\infty} (-1)^k \varepsilon^k p^{k/2} \varkappa^k N_k(y)\Big) e^{-\sqrt{p}\varkappa x} \end{aligned} \tag{2.13}$$

The Wronskian is given by the formula

$$W(z^{(1)}, z^{(2)}) = -\frac{\varkappa\sqrt{p}}{A(y)}\Big(\frac{1}{\langle A^{-1}\rangle} + \sum_{m=1}^{\infty} \varepsilon^{2m} p^m \varkappa^{2m} d_m\Big) \tag{2.14}$$

where d_m ($m = 1, 2, 3, \ldots$) are constant numbers given by the following expressions:

$$d_m = A(y)\Big\{\sum_{n=0}^{m} N_{2m-2n}\Big(\frac{dN_{2n+1}}{dy} + N_{2n}\Big) - \sum_{n=0}^{m-1} N_{2m-2n-1}\Big(\frac{dN_{2n+2}}{dy} + N_{2n+1}\Big)\Big\} \tag{2.15}$$

In particular, the constants N_{k0} may be chosen in such a way that

$$d_m = 0 \quad (m = 1, 2, 3, \ldots) \tag{2.16}$$

and in that case, instead of (2.14), we have the simple formula

$$W(z^{(1)}, z^{(2)}) = -\varkappa \sqrt{p}/(A(y)\langle A^{-1}\rangle) \tag{2.17}$$

Note that formulas (2.15) and (2.16) uniquely define the constants N_{k0}, so that these conditions, supplementing the local problems (2.10) and (2.11), uniquely define the functions $N_k(y)$ ($k = 1, 2, 3, \ldots$).

By Lemma 2, the general solution of the non-homogeneous equation (2.5) may be written as

$$z(x, p) = \frac{\langle A^{-1}\rangle}{\varkappa} \int_0^\infty \rho^{(\varepsilon)}(\xi) f(\xi) G(x, \xi, p)\, d\xi \tag{2.18}$$

$$G(x, \xi, p) = \begin{cases} z^{(2)}(x, p)\, z^{(1)}(\xi, p)\, p^{-1/2}, & \xi < x \\ z^{(1)}(x, p)\, z^{(2)}(\xi, p)\, p^{-1/2}, & \xi > x \end{cases}$$

We now return from the Laplace transform (2.18) to the source function, carry out some reduction and set $t = 0$ [see (2.3)], to obtain the proof of the following theorem, which solves our problem of expanding a function in terms of solutions of Eq. (2.1).

Theorem 1. Any piecewise-smooth function can be expressed as an integral transform as follows:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty z_c^{(\varepsilon)}(x, \mu) F_c(\mu)\, d\mu \tag{2.19}$$

where

$$F_c(\mu) = \sqrt{\frac{2}{\pi}} \left(\frac{\langle A^{-1}\rangle}{\langle\rho\rangle}\right)^{1/2} \int_0^\infty \rho^{(\varepsilon)}(\xi) f(\xi)\, z_c^{(\varepsilon)}(\xi, \mu)\, d\xi \tag{2.20}$$

$$z_c^{(\varepsilon)}(x, \mu) = \left[1 + \sum_{k=1}^\infty \varepsilon^k N_k\left(\frac{x}{\varepsilon}\right) \frac{d^k}{dx^k}\right] \cos(\varkappa\mu x) \tag{2.21}$$

$N_k(x/\varepsilon)$ are singly periodic functions of $y = x/\varepsilon$, which are solutions of the recursive chain of local problems (2.10) and (2.11) with conditions (2.15) and (2.16)—the latter determine N_{k0}. The function $z_c^{(\varepsilon)}(x, \mu)$ is a solution of Eq. (2.1).

Remark 1. The local problems (2.10) and (2.11) are ordinary differential equations and are solvable by quadratures. In particular,

$$N_1(y) = -y + \frac{1}{\langle A^{-1}\rangle} \int_0^y A^{-1}(\xi)\, d\xi + N_{10}$$

$$N_2(y) = -y^2/2 + y/2 + (1/2 + N_{10} - y)(N_1(y) - N_{10}) -$$

$$-\varkappa^{-2}\left\{(N_1(y) - N_{10} + y)\left(\int_0^1 A^{-1}(\eta)\, d\eta \int_0^\eta \rho(\xi)\, d\xi\right) - \int_0^y A^{-1}(\eta)\, d\eta \int_0^\eta \rho(\xi)\, d\xi\right\} + N_{20}$$

Remark 2. Conditions (2.15) and (2.16) may be written as a recursive system of algebraic equations in the constants N_{k0} ($k = 1, 2, 3, \ldots$):

$$d_1 = N_{20}C_1 + C_3 - N_{10}C_2 = 0$$
$$d_2 = N_{40}C_1 + N_{20}C_3 + C_5 - N_{30}C_2 - N_{10}C_4 = 0 \tag{2.22}$$
$$d_3 = N_{60}C_1 + N_{40}C_3 + N_{20}C_5 + C_7 - N_{50}C_2 - N_{30}C_4 - N_{10}C_6 = 0, \ldots$$

where

$$C_k = (A dN_k/dy + AN_{k-1})|_{y=0}$$

In particular, we can set

$$N_{20} = N_{40} = N_{60} = \ldots = 0 \tag{2.23}$$

and define $N_{10}, N_{30}, \ldots$ by

$$N_{10} = C_3/C_2, \quad N_{30} = C_5/C_2 - C_3C_4/C_2^2 \tag{2.24}$$
$$N_{50} = C_7/C_2 - C_4C_5/C_2^2 + C_3C_4^2/C_2^3 - C_3C_6/C_2^2, \ldots$$

If that is done it follows from (2.21) that

$$z_c^{(\varepsilon)}(x, \mu)|_{x=0} = 1 \tag{2.25}$$

Remark 3. In the limiting special case when Eq. (2.1) has constant coefficients,

$$A^{(\varepsilon)}(x) = A = \text{const}, \quad \rho^{(\varepsilon)}(x) = \rho = \text{const}$$

Equations. (2.6) and (2.9) are the same and by solving the local problems (2.10) under conditions (2.15) and (2.16), we obtain

$$N_k(y) \equiv 0, \quad k = 1, 2, 3, \ldots \tag{2.26}$$

It follows from (2.21) that

$$z_c^{(\varepsilon)}(x, \mu) = \cos(\varkappa\mu x)$$

and if $A = \rho$ (respectively, $\varkappa = 1$) the integral transform (2.19) and (2.20) is simply the Fourier cosine transform. Thus, (2.19)–(2.21) is a generalization of the Fourier cosine transform to the case in which the coefficients of Eq. (2.1) are rapidly oscillating ε-periodic functions.

3. To obtain an analogue of the Fourier sine transform, we take the linearly independent particular solutions of Eq. (2.9) to be

$$z_0^{(1)}(x, p) = \operatorname{sh}(\sqrt{p}\varkappa x), \quad z_0^{(2)}(x, p) = e^{-\sqrt{p}\varkappa x} \tag{3.1}$$

Here we can prove the following theorem.

Theorem 2. For any piecewise-smooth function $f(x)$ one has the following integral transform:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty z_s^{(\varepsilon)}(x, \mu) F_s(\mu)\, d\mu \tag{3.2}$$

$$F_s(\mu) = \sqrt{\frac{2}{\pi}} \left(\frac{\langle A^{-1}\rangle}{\langle\rho\rangle}\right)^{1/2} \int_0^\infty \rho^{(\varepsilon)}(\xi) f(\xi) z_s^{(\varepsilon)}(\xi, \mu)\, d\xi \tag{3.3}$$

$$z_s^{(\varepsilon)}(x, \mu) = \left[1 + \sum_{k=1}^\infty \varepsilon^k N_k\left(\frac{x}{\varepsilon}\right)\frac{d^k}{dx^k}\right] \sin(\varkappa\mu x) \tag{3.4}$$

where $N_k(x/\varepsilon)$ are singly periodic functions of $y = x/\varepsilon$, which are solutions of the recursive chain of local problems (2.10) and (2.11), the constants N_{k0} being determined by conditions (2.15) and (2.16). The function $z_s^{(\varepsilon)}(x, \mu)$ is a solution of Eq. (2.1).

Remark 4. In the solution of the algebraic equations (2.22), instead of (2.23) and (2.24) one can set

$$N_{10} = N_{30} = N_{50} = \ldots = 0$$

and determine $N_{20}, N_{40}, \ldots$ from the equations

$$N_{20} = -C_3/C_1, \quad N_{40} = C_3^2/C_1^2 - C_5/C_1, \ldots$$

In that case the expansion (3.4) gives

$$z_s^{(\varepsilon)}(x, \mu)_{x=0} = 0 \tag{3.5}$$

Remark 5. In the limiting special case

$$A^{(\varepsilon)}(x) = \rho^{(\varepsilon)}(x) = \text{const}$$

we conclude, as in Remark 4, that the transform (3.2)–(3.4) is simply the Fourier sine transform.

4. We will now use the above integral transforms to solve problems (1.1) and (1.4) with condition (1.2) on the boundary of a half-plane. Using (2.19), we will seek the solution in the form

$$u = \sqrt{\frac{2}{\pi}} \int_0^\infty z_c^{(\varepsilon)}(x_1, \mu) U_c(x_2, \mu) \, d\mu \tag{4.1}$$

where $z_c^{(\varepsilon)}(x_1, \mu)$ is a solution of Eq. (2.1) with

$$A^{(\varepsilon)}(x) = \lambda_1(y), \quad \rho^{(\varepsilon)}(x) = \lambda_2(y) \tag{4.2}$$

Substituting (4.1) into (1.1) and using (2.1) and (4.2), we obtain an equation for the function $U_c(x_2, \mu)$:

$$\partial^2 U_c(x_2, \mu)/\partial x_2^2 - \mu^2 U_c(x_2, \mu) = 0 \tag{4.3}$$

A solution of this equation which is bounded as a function of x_2 is

$$U_c(x_2, \mu) = C(\mu) e^{-\mu x_2} \tag{4.4}$$

Substituting (4.4) into (4.1), setting $x_2 = 0$ and using (1.2), we obtain

$$\sqrt{\frac{2}{\pi}} \int_0^\infty z_c^{(\varepsilon)}(x_1, \mu) C(\mu) \, d\mu = g(x_1) \tag{4.5}$$

Returning in (4.5) to the source function of the transform (2.20), we find that

$$C(\mu) = \sqrt{\frac{2}{\pi}} \left(\frac{\langle\lambda_1^{-1}\rangle}{\langle\lambda_2\rangle}\right)^{1/2} \int_0^\infty \lambda_2^{(\varepsilon)}(\xi) g(\xi) z_c^{(\varepsilon)}(\xi, \mu) \, d\xi \tag{4.6}$$

Substituting (4.4) and (4.5) into (4.1), we get the required solution in quadratures:

$$u(x_1, x_2) = \frac{2}{\pi} \left(\frac{\langle\lambda_1^{-1}\rangle}{\langle\lambda_2\rangle}\right)^{1/2} \int_0^\infty z_c^{(\varepsilon)}(x_1, \mu) e^{-\mu x_2} \, d\mu \int_0^\infty \lambda_2^{(\varepsilon)}(\xi) g(\xi) z_c^{(\varepsilon)}(\xi, \mu) \, d\xi$$

The treatment of problems (1.1) and (1.4) with condition (1.3) is analogous. We then obtain

$$u(x_1, x_2) = \frac{2}{\pi}\left(\frac{\langle\lambda_1^{-1}\rangle}{\langle\lambda_2\rangle}\right)^{1/2} \int_0^\infty z_c^{(\varepsilon)}(x_1, \mu) e^{-\mu x_2}/\mu \, d\mu \int_0^\infty q(x_1) z_c^{(\varepsilon)}(\xi, \mu) \, d\mu$$

Note that the method will also work for boundary-value problems with mixed conditions on the boundary.

5. We will now obtain a fundamental solution of Eq. (1.1) in the case when

$$\lambda_1^{(\varepsilon)}(x_1) = \lambda_2^{(\varepsilon)}(x_1) = \lambda(y)$$

i.e. we shall use the integral transforms (2.19)–(2.21) to solve the equation

$$\frac{\partial}{\partial x_1}\left(\lambda(y)\frac{\partial u}{\partial x_1}\right) + \lambda(y)\frac{\partial^2 u}{\partial x_2^2} = \delta(x_1)\delta(x_2) \tag{5.1}$$

The solution will be sought in the form

$$u = \sqrt{\frac{2}{\pi}} \int_0^\infty U(x_2, \mu) z_c^{(\varepsilon)}(x_1, \mu) \, d\mu \tag{5.2}$$

substituting $A^{(\varepsilon)}(x_1) = \rho^{(\varepsilon)}(x_1) = \lambda(y)$ into (2.1) and (2.20).

In view of (2.19), (2.20) and (2.25), we obtain the following representation for the δ-function:

$$\lambda^{-1}(y)\delta(x_1) = \frac{1}{\pi}\left(\frac{\langle\lambda^{-1}\rangle}{\langle\lambda\rangle}\right)^{1/2} \int_0^\infty z_c^{(\varepsilon)}(x_1, \mu) \, d\mu \tag{5.3}$$

Substituting (5.2) and (5.3) into (5.1), we get

$$\frac{\partial^2 U(x_2, \mu)}{\partial x_2^2} - \mu^2 U(x_2, \mu) = \frac{1}{\sqrt{2\pi}}\left(\frac{\langle\lambda^{-1}\rangle}{\langle\lambda\rangle}\right)^{1/2} \delta(x_2) \tag{5.4}$$

The solution of Eq. (5.4) is

$$U(x_2, \mu) = -\frac{1}{2\sqrt{2\pi}}\left(\frac{\langle\lambda^{-1}\rangle}{\langle\lambda\rangle}\right)^{1/2} \frac{e^{-\mu|x_2|}}{\mu} \tag{5.5}$$

and then

$$u = -\frac{1}{2\pi}\left(\frac{\langle\lambda^{-1}\rangle}{\langle\lambda\rangle}\right)^{1/2} \int_0^\infty z_c^{(\varepsilon)}(x_1, \mu) \frac{e^{-\mu|x_2|}}{\mu} \, d\mu$$

In view of the representation (2.21), this expression may be written in the form

$$u = -\frac{1}{2\pi}\left(\frac{\langle\lambda^{-1}\rangle}{\langle\lambda\rangle}\right)^{1/2} \left[1 + \sum_{k=1}^\infty \varepsilon^k N_k(x_1/\varepsilon) \frac{\partial^k}{\partial x_1^k}\right] \int_0^\infty \frac{e^{-\mu|x_2|}}{\mu} \cos(\varkappa\mu x_1) \, d\mu$$

Since

$$\int_0^\infty \frac{e^{-\mu|x_2|}}{\mu} \cos(\varkappa\mu x_1) \, d\mu = -\ln\left(\sqrt{x_2^2 + \varkappa^2 x_1^2}\right),$$

the required fundamental solution of Eq. (5.1) is

$$u = \frac{1}{2\pi}\left(\frac{\langle\lambda^{-1}\rangle}{\langle\lambda\rangle}\right)^{1/2} \left[1 + \sum_{k=1}^\infty \varepsilon^k N_k(x_1/\varepsilon) \frac{\partial^k}{\partial x_1^k}\right] \ln\left(\sqrt{x_2^2 + \varkappa^2 x_1^2}\right) \tag{5.6}$$

In the limiting special case when $\lambda(y)\equiv 1$, relations (2.26) and formula (5.6) give the well-known expression for the fundamental solution of the two-dimensional Laplace equation.

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THE EIGENMODE EXPANSION METHOD FOR OSCILLATIONS OF AN ELASTIC BODY WITH INTERNAL AND EXTERNAL FRICTION†

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A method is proposed for solving dynamical problems for a viscoelastic body (the Kelvin–Voigt model) in a massless viscous medium. Interaction with the external medium produces on the boundary of the body stresses proportional to the rate of displacement. The model of external friction is that used for modelling dynamical processes in elastic media filling an infinite domain [1, 2]. The implementation of numerical methods of solution requires an equivalent restatement of the problem in a finite domain, using external viscous friction to allow for the radiation of energy at infinity.

FROM THE mathematical point of view, the eigenvalue spectral problem in the presence of friction is not self-adjoint and the eigenfunctions are not orthogonal. For an elastic body with friction, the

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